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2003 J. Phys.: Condens. Matter 15 6969

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Exact transmission moments in one-dimensional weak localization and single-parameter scaling

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Received 11 June 2003

Published 3 October 2003

Online at stacks.iop.org/JPhysCM/15/6969

Abstract

We obtain for the first time the expressions for the mean and the variance of the transmission coefficient for an Anderson chain in the weak localization regime, using exact expansions of the complex transmission and reflection coefficients to fourth order in the weakly disordered site energies. These results confirm the validity of single-parameter scaling theory in a domain where the higher transmission cumulants may be neglected. We compare our results with earlier results for transmission cumulants in the weak localization domain based on the phase randomization hypothesis.

1. Introduction

The validity of the single-parameter scaling (SPS) hypothesis in the scaling theory of localization of Abrahams *et al* [1] has been the object of numerous discussions since its publication in 1979. This scaling theory assumes that the scaling of a typical conductance g as a function of size L of a d -dimensional disordered sample is a function of only one parameter, namely the typical conductance itself.

Soon after the appearance of [1] it became clear that one should consider the scaling of the full probability distribution of conductance (or the distribution of a related scaling variable) rather than simply that of a typical conductance [2]. SPS then requires that the distribution of the chosen variable should scale to a universal form depending only on one parameter. A simple example of such a statistical distribution is the familiar normal distribution in which the large scale dispersion σ^2 decays to zero via the law of large numbers relating it to the growth rate of the mean value, the only parameter of this distribution.

Many aspects of scaling of distributions of transport related quantities such as the resistance $r = 1/g$ were discussed some years ago by Shapiro and co-workers [3] and in a more recent review [4].

Until recently analytical discussions of SPS have invariably relied on the so-called phase randomization hypothesis [2]. This hypothesis assumes the existence of a microscopic length scale, shorter than the localization length, over which the phases of the complex reflection

and transmission coefficients of a disordered sample are randomized in such a way that their distribution becomes uniform.

Let us briefly recall the scaling results for a 1D-disordered chain of length L . In this case a convenient scaling variable related to the conductance is the finite scale Lyapunov exponent [2]

$$\begin{aligned}\bar{\gamma}_L &= \frac{1}{2L} \ln \left(1 + \frac{1}{g} \right) \\ &= -\frac{1}{2L} \ln t_L,\end{aligned}\quad (1)$$

where the second line follows from using the Landauer four-probe formula, $r = 1/g = r_L/t_L$, which relates the resistance r (expressed in natural units of $\pi\hbar/e^2$) to the transmission (t_L) and reflection (r_L) coefficients of the chain. The advantage of the scaling variable $\bar{\gamma}_L$ is that under the assumption of complete phase randomization it obeys a normal distribution [3, 4] of the type alluded to above, in the strong localization regime, $L \gg L_c$ (where $L_c^{-1} \equiv \gamma = \lim_{L \rightarrow \infty} \gamma_L$ is the inverse localization length), for weak disorder¹. The distribution is centred at the asymptotic value L_c^{-1} , with a dispersion

$$\sigma^2 = \text{var } \bar{\gamma}_L = \frac{\gamma}{L}, \quad (2)$$

i.e. it describes SPS in terms of $\gamma = L_c^{-1}$. Equation (2) coincides with the general large number law property for the validity of SPS for $\bar{\gamma}_L$ proposed recently by Deych *et al* for the general case where phase randomization is not hypothesized. In their work [7] Deych *et al* studied the condition (2) analytically for an Anderson chain with a Cauchy distribution of site energies. This has led them to identify a new characteristic length beyond which SPS holds true in their system [7]. On the other hand, under the random phase assumption one-parameter scaling also exists in the weak localization (quasi metallic) regime for weak disorder [3, 4], where $L \ll L_c$. In this case the resistance $r \ll 1$ ($r \simeq 2L\gamma$) follows an exponential (Poisson) distribution [3, 4] depending on the sole parameter L_c^{-1} . The condition for one-parameter scaling analogous to (2) then reads (see also section 3)

$$\sigma^2 = \langle \bar{\gamma}_L \rangle^2, \quad (3)$$

where $\langle \dots \rangle$ denotes averaging over the disorder. Equation (3) implies one-parameter scaling in a domain where the higher cumulants of the resistance distribution are subdominant. We also note that (3) is exactly verified (with $\langle \bar{\gamma}_L \rangle^2 = \gamma^2$) by the exponential distribution of resistance obtained in the framework of the random phase assumption for $L \ll L_c$.

The purpose of this paper is to analyse the validity of one-parameter scaling theory in the weak localization regime, by deriving exact analytical results for the mean value and variance of the transmission coefficient t_L for a weakly disordered Anderson chain, without using phase randomization. The resulting expressions for the variance σ^2 and the mean value $\langle \bar{\gamma}_L \rangle$ of the variable $\bar{\gamma}_L$ are shown to obey approximately (for $L \gg a$, with a the lattice parameter) a universal relation analogous to (3). Here the mean $\langle \bar{\gamma}_L \rangle$, which turns out to depend on L , is completely defined in terms of a fixed localization length. The complex transmission coefficient T_L ($t_L = |T_L|^2$) and reflection coefficient R_L ($r_L = |R_L|^2$) are obtained in section 2 by iterating the exact coupled recursion relations obeyed by these quantities [8, 9], for weak disorder in the regime $L \ll L_c$. These results are used for obtaining explicit forms of the first

¹ Note that the second line of equation (1) follows also from the definition of the alternative scaling variable $\gamma'_L = -(1/2L) \ln g$, with g given by the Landauer two-probe conductance: $g = t_L$. This Lyapunov exponent is also self-averaging and obeys a normal distribution in the localized regime for random phases. As is well-known the Landauer two-probe resistance, t_L^{-1} , differs from the Landauer four-probe expression, r_L/t_L , by a fixed contact resistance, which implies, in particular, that the resistance of a perfect conductor is finite [5, 6].

two moments of the transmission coefficient t_L and of the corresponding Lyapunov exponent $\tilde{\gamma}_L$. These moments are discussed in connection with the validity of one-parameter scaling theory in section 3, where they are also compared with previous results based on the random phase assumption.

2. Transmission in the weak localization regime

Consider a disordered linear chain of length $L = Na$ composed of N disordered one-orbital sites of spacing $a = 1$ described by the Schrödinger equation

$$\varphi_{n+1} + \varphi_{n-1} + \varepsilon_n \varphi_n = E \varphi_n, \quad (4)$$

where φ_m is the wavefunction amplitude and ε_m the random energy of the site m measured (like E also) in units of the constant hopping rate between nearest-neighbour sites. We assume the ε_m to be identically distributed independent Gaussian variables with mean zero and correlation

$$\langle \varepsilon_m \varepsilon_n \rangle = \varepsilon_0^2 \delta_{m,n}. \quad (5)$$

The usual Anderson model of uniformly distributed site energies between $-W$ and W correspond to $\varepsilon_0^2 = W^2/12$. The disordered chain is connected to electron reservoirs by semi-infinite non-disordered leads defined by sites $m > N$ and $m < 1$, respectively. As indicated in section 1, we wish to calculate the transmission coefficient $t_N = |T_N|^2$ and to study its statistical moments in the weak localization regime $L \ll L_c$, for weak disorder. The inverse localization length is defined by

$$\gamma = \frac{1}{L_c} = - \lim_{N \rightarrow \infty} \frac{1}{2N} \ln t_N. \quad (6)$$

The complex transmission and reflection amplitudes for a plane wave incident from the right ($n > N$) with wavenumber k and energy

$$E = 2 \cos k, \quad 0 \leq k \leq \pi, \quad (7)$$

obey the exact coupled recursion relations [8, 9]

$$T_N = \frac{e^{ik} T_{N-1}}{1 - i v_N (1 + e^{2ik} R_{N-1})}, \quad (8)$$

$$R_N = \frac{e^{2ik} R_{N-1} + i v_N (1 + e^{2ik} R_{N-1})}{1 - i v_N (1 + e^{2ik} R_{N-1})}, \quad (9)$$

$$v_N = \frac{\varepsilon_N}{2 \sin k}, \quad (10)$$

which relate the amplitudes $T_N(R_N)$ for the chain of N disordered sites to those for a chain of $N - 1$ sites, with the initial conditions $T_0 = 1$ and $R_0 = 0$. The choice of wavenumbers in (7) is such that the leads eigenfunctions $\varphi_n \sim e^{\pm i k n}$ correspond to Bloch waves travelling from left to right and from right to left, respectively. The solution of (9) may be readily obtained in terms of the reflection amplitudes for chains whose lengths are comprised between 1 and $N - 1$:

$$T_N = e^{ikN} \prod_{n=1}^N \frac{1}{1 - i v_n (1 + e^{2ik} R_{n-1})}. \quad (11)$$

For weak disorder we expand (11) to successive orders in the site energies ε_n . As will be readily seen below, the validity of this perturbation expansion is restricted typically to chain lengths $L \ll L_c$ (the weak localization regime), where L_c is the localization length for weak disorder

first derived by Thouless [10]. The study of the dispersion of the transmission coefficient requires the inclusion of terms up to fourth order in the expansion of (11). We thus obtain

$$T_N = e^{ikN} \left[1 + \sum_{m=1}^N (P_m - 1) + \sum_{\substack{m,n=1 \\ m \neq n}}^N (P_m - 1)(P_n - 1) + \sum_{\substack{m,n,p \\ m \neq n \neq p \neq m}} (P_m - 1)(P_n - 1)(P_p - 1) + \sum_{\substack{m,n,p,q \\ m \neq n \neq p \neq q \\ n \neq q \neq m \neq p}} (P_m - 1)(P_n - 1)(P_p - 1)(P_q - 1) \right], \quad (12)$$

$$P_j = \frac{1}{1 + x_j}, \quad x_j = -iv_j(1 + e^{2ik} R_{j-1}).$$

In each one of the successive terms in (12), we then retain contributions up to fourth order in the variables v_j , via the expansion of the P_j in power of x_j and corresponding expansions of R_j in successive order contributions in the reflection amplitudes,

$$R_n = \sum_{q=1}^4 R_n^{(q)}. \quad (13)$$

The contributions $R_n^{(q)}$ at successive orders q are determined by the perturbation equations obtained by expanding the recursion equation analogous to (9) relating R_n and R_{n-1} for systems of length $1 \leq n \leq N - 1$ and $n - 1$:

$$R_n^{(1)} = e^{2ik} R_{n-1}^{(1)} + iv_n, \quad (14a)$$

$$R_n^{(2)} = e^{2ik} R_{n-1}^{(2)} + 2iv_n e^{2ik} R_{n-1}^{(1)} - v_n^2, \quad (14b)$$

$$R_n^{(3)} = e^{2ik} R_{n-1}^{(3)} + 2iv_n e^{2ik} R_{n-1}^{(2)} + iv_n e^{4ik} R_{n-1}^{(1)2} - 2v_n^2 e^{2ik} R_{n-1}^{(1)} - iv_n^3, \quad (14c)$$

whose solutions are

$$R_n^{(1)} = i \sum_{p=1}^n v_p e^{2ik(n-p)}, \quad (15a)$$

$$R_n^{(2)} = - \sum_{p=1}^n v_p e^{2ik(n-p)} \left[v_p + 2 \sum_{q=1}^{p-1} v_q e^{2ik(p-q)} \right], \quad (15b)$$

$$R_n^{(3)} = -i \sum_{p=1}^n v_p e^{i2(n-p)} \left[v_p^2 - 2iv_p R_{p-1}^{(1)} e^{2ik} - R_{p-1}^{(1)2} e^{4ik} - 2R_{p-1}^{(2)} e^{2ik} \right]. \quad (15c)$$

Next, by using (12) and (13) and (15a)–(15c), we express the transmission coefficient $t_N = |T_N|^2$ to fourth order in the site energies and, finally, we find its average $\langle t_N \rangle$ using (5) together with the factorization property for independent Gaussian variables,

$$\langle \varepsilon_m \varepsilon_n \varepsilon_p \varepsilon_q \rangle = \langle \varepsilon_m \varepsilon_n \rangle \langle \varepsilon_p \varepsilon_q \rangle + \langle \varepsilon_m \varepsilon_p \rangle \langle \varepsilon_n \varepsilon_q \rangle + \langle \varepsilon_m \varepsilon_q \rangle \langle \varepsilon_n \varepsilon_p \rangle, \quad (16)$$

which is valid for arbitrary indices m, n, p, q different or not [11].

These straightforward but rather tedious calculations yield the final result

$$\langle t_N \rangle = 1 - \frac{2N}{L_c} - \frac{2N^2}{L_c^2} \left[5 - \frac{1}{N} - \frac{2}{N^2} \frac{\sin^2 kN}{\sin^2 k} \left(3 + \frac{\cos^2 kN}{\cos^2 k} \right) \right] + O(\varepsilon_0^6), \quad (17)$$

which is exact to quadratic order in ε_0^2 . Here

$$\frac{1}{L_c} = \frac{\varepsilon_0^2}{8 \sin^2 k} = \frac{\varepsilon_0^2}{2(4 - E^2)}, \tag{18}$$

is the Thouless expression [10] for the localization length, which is valid to lowest order in the weak disorder. The perturbation expression (17) shows that besides the smallness of ε_0^2 its validity requires typically $N \ll L_c$, which implies that the eigenstates are delocalized on the scale of the chain length. We further note that while the lowest correction in $\langle t_N \rangle$ for weak disorder is of order N/L_c , the next higher contribution proportional to ε_0^4 includes three types of terms respectively proportional to $(N/L_c)^2$, $(N/L_c)\varepsilon_0^2$ and ε_0^4 . By a similar calculation of the second moment, $\langle t_N^2 \rangle$, we obtain

$$\langle t_N^2 \rangle = 1 - \frac{4N}{L_c} + \frac{4N^2}{L_c^2} \left[2 - \frac{2}{N} + \frac{1}{N^2} \frac{\sin^2 kN}{\sin^2 k} \left(4 + \frac{2 \cos^2 kN}{\cos^2 k} - \frac{\sin^2 kN}{2 \sin^2 2k} \right) \right] + O(\varepsilon_0^6). \tag{19}$$

The calculation of $\langle t_N^2 \rangle$ has been greatly facilitated by making use of the expression of t_N to quartic order and of the corresponding average (17). From (19) and (18) we finally get

$$\text{var } t_N = \frac{12N^2}{L_c^2} \left[2 - \frac{1}{N} - \frac{1}{3N^2} \frac{\sin^2 kN}{\sin^2 k} \left(2 + \frac{\sin^2 kN}{2 \sin^2 2k} \right) \right] + O(\varepsilon_0^6). \tag{20}$$

The first and second moments and the variance of $\bar{\gamma}_L$ in (1) are easily found from (17) and (19) by expanding $\ln t_N$ in powers of the small quantity $1 - t_N$. Thus we obtain

$$\begin{aligned} \langle \bar{\gamma}_N \rangle &= -\frac{1}{2N} \langle \ln t_N \rangle = \frac{1}{L_c} \left(1 + \frac{4N}{L_c} \left[3 - \frac{1}{N} - \frac{1}{2N^2} \frac{\sin^2 kN}{\sin^2 k} \right. \right. \\ &\quad \left. \left. \times \left(4 + \frac{\cos^2 kN}{\cos^2 k} + \frac{\sin^2 kN}{4 \sin 2k} \right) \right] \right) + O(\varepsilon_0^6), \end{aligned} \tag{21}$$

$$\langle \bar{\gamma}_N^2 \rangle = \frac{1}{L_c^2} \left[1 - \frac{3}{N} - \frac{1}{N^2} \frac{\sin^2 kN}{\sin^2 k} \left(2 + \frac{\sin^2 kN}{2 \sin 2k} \right) \right] + O(\varepsilon_0^6), \tag{22}$$

$$\sigma^2 = \text{var } \bar{\gamma}_N = \frac{3}{L_c^2} \left[2 - \frac{1}{N} - \frac{1}{N^2} \frac{\sin^2 kN}{\sin^2 k} \left(2 + \frac{\sin 2kN}{2 \sin 2k} \right) \right]. \tag{23}$$

By ignoring the terms of orders $1/N$ and $1/N^2$ relative to unity in the square brackets of (21)–(23) it thus follows that

$$\sigma^2 = 6(\langle \bar{\gamma}_N \rangle)^2 + O(\varepsilon_0^6). \tag{24}$$

3. Discussion and concluding remarks

Equation (24) is a universal relation expressing the variance of the scaling variable $\bar{\gamma}_N$ in terms of the mean, which is itself defined in terms of the Thouless localization length (18) alone, when terms of order $1/N$ and $1/N^2$ are neglected relative to unity in (21). It is natural to expect that, within the same approximation, similar universal forms in terms of $\langle \bar{\gamma}_N \rangle$ exist for the higher fluctuation cumulants of $\bar{\gamma}_N$. Thus our work suggests that the exact distribution of $\bar{\gamma}_N$ (which does not rely on phase randomization) in the weak localization regime obeys SPS. Note that (24) differs by a factor of 6 from the universal relation (3) for the weak localization regime obtained under the phase randomization assumption. We observe in passing that the mean of $\bar{\gamma}_N$ in (21) differs from the Thouless inverse localization lengths (18). This is because the inverse localization length is, by definition, the non-random central limit of $\bar{\gamma}_N$. In order to obtain L_c^{-1} from our analysis for $N \ll L_c$ one has to let $|\varepsilon_0| \rightarrow 0$ before taking the large N limit. In this limit only the leading order term of the weak disorder expansion of $\langle t_N \rangle$ in (17) is

relevant for finding L_c^{-1} . We believe that the reason why this term yields the correct (Thouless) localization length in spite of the fact $\bar{\gamma}_N$ is not self-averaging in the domain $N \ll L_c$ is because the non-self averaging is only marginal, i.e. $\lim_{N \rightarrow \infty} \lim_{|\varepsilon_0| \rightarrow 0} \frac{\sigma^2}{\langle \bar{\gamma}_N \rangle^2} = \text{constant}$ in this case.

The aim of the present work has been to discuss the scaling of $\bar{\gamma}_L$ in the weak localization regime without using the phase randomization hypothesis. In this context we now compare the above results for the transmission coefficient moments with earlier results [12] obtained from an invariant imbedding model [13, 14] analysed by assuming phase randomization [12]. The coupled invariant imbedding equations for the complex reflection (R_N) and transmission coefficients (T_N) [13, 14] have since been shown [8] to correspond to the continuum limit of the Schrödinger equation for the one-dimensional Anderson model for weak disorder. This makes the comparison of the present results with those of [12] the more relevant. Under the random phase assumption we obtained the following exact results² for the transmission coefficient moments for $L \ll L_c$:

$$\langle t_N^m \rangle = \left(1 + 2m \frac{N}{L_c} \right)^{-1}, \quad m = 0, 1, 2, 3, \dots, \quad (25)$$

which leads to the distribution of the transmission coefficient [12]

$$P_t(t_N) = \frac{L_c}{2N} t^{(L_c/2N-1)}, \quad (26)$$

which depends on the single scale parameter L_c . Note that the exact moments (17) and (19) which do not rely on the random phase assumption reveal important deviations from equation (25) at order ε_0^4 , even when the terms in $1/N$ and $1/N^2$ in the square brackets are neglected³. The exact random phase expression for $\text{var } t_N$,

$$\text{var } t_N = \frac{4N^2}{L_c^2 \left(1 + \frac{4N}{L_c} \right) \left(1 + \frac{2N}{L_c} \right)^2}, \quad (27)$$

obtained from (25) differs also from the exact expression (20) to order N^2/L_c^2 . For the sake of completeness we finally list the expressions for the first and second moments and the variance of $\bar{\gamma}_N$ obtained from expanding the logarithm of (25) for $m = 1$ and 2 through order ε_0^4 :

$$\langle \bar{\gamma}_N \rangle = \frac{1}{L_c} \left(1 - \frac{2N}{L_c} \right), \quad (28)$$

$$\langle \bar{\gamma}_N^2 \rangle = \frac{4}{L_c^2}, \quad (29)$$

$$\sigma^2 = (\langle \bar{\gamma}_N \rangle)^2. \quad (30)$$

References

- [1] Abrahams E, Anderson P W, Licciardello D C and Ramakrishnan T V 1979 *Phys. Rev. Lett.* **42** 673
 [2] Anderson P W, Thouless D J, Abrahams E and Fisher D S 1980 *Phys. Rev. B* **22** 3519

² We recall that the derivation of [8] leads to a natural definition of the localization length in the framework of the invariant imbedding model: it is simply given by the continuum limit of the localization length (18). This is twice the localization length which was used in [12] (which corresponded to a different definition of the mean free path in the classical resistivity [2] in terms of parameters of the invariant imbedding model) and leads to the coefficient 2 on the rhs of (25).

³ Using the Landauer formula, $r = t^{-1} - 1$, the distribution (26) was shown in [12] to yield the familiar exponential distribution of resistance for $\rho \ll 1$ ($L \ll L_c$) [3]. Actually, however, this derivation yields the slightly more accurate expression $P_N(\rho) = t^{-1} \exp[-(t^{-1} + 1)\rho]$, $\ell = 2N/L_c$. The improved expression turns out to be necessary for recovering (26) and (25), starting from the distribution of resistance.

- [3] Shapiro B 1987 *Phil. Mag.* B **56** 1031
Cohen A, Roth Y and Shapiro B 1988 *Phys. Rev. B* **38** 12125
- [4] Janssen M 1998 *Phys. Rep.* **295** 1
- [5] Imry Y 1997 *Introduction to Mesoscopic Physics* (Oxford: Oxford University Press)
- [6] Datta S 1995 *Electronic Transport in Mesoscopic Systems* (Cambridge: Cambridge University Press)
- [7] Deych L I, Lisansky A A and Altshuler B L 2000 *Phys. Rev. Lett.* **84** 2678
Deych L I, Lisansky A A and Altshuler B L 2001 *Phys. Rev. B* **64** 224202
- [8] Heinrichs J 2002 *Phys. Rev. B* **65** 075112
- [9] Barnes C and Luck J M 1990 *J. Phys. A: Math. Gen.* **23** 1717
- [10] Thouless D J 1979 *Ill-Condensed Matter* ed R Balian, R Maynard and G Toulouse (Amsterdam: North-Holland)
- [11] See, for example, Van Kampen N G 1981 *Stochastic Processes in Physics and Chemistry* (Amsterdam: North-Holland)
- [12] Heinrichs J 1989 *J. Phys.: Condens. Matter* **1** 2651
- [13] Bellman R and Wing C M 1976 *An Introduction to Invariant Imbedding* (New York: Wiley)
- [14] For a review of invariant imbedding and applications, see Rammal R and Douçot B 1987 *J. Physique* **48** 509